DERIVATIVES (examples - part III)

Derivatives of logarithmic functions

If we have $y = f(x)$, where is $y > 0$ and $y \neq 1$, then:

$$(\ln y)` = \frac{y`}{y} = \frac{f`(x)}{f(x)}$$

Example 1.

Find derivative of function: $\mathbf{y = x^x}$

Solution:

$y = x^x$

$\ln y = \ln x^x$ **we will use rule: $\ln A^n = n \ln A$**

$\ln y = x \ln x$ **now differentiate**

$$\frac{y`}{y} = x` \ln x + (\ln x)` x$$

$$\frac{y`}{y} = \ln x + \frac{1}{x} x$$

$$\frac{y`}{y} = \ln x + 1$$ **all multiply with y**

$y` = y(\ln x + 1)$ **replace y with x^x**

$y` = x^x(\ln x + 1)$ the final solution!

Example 2.

Find derivative of function: $y = (\cos x)^{\sin x}$

Solution:

$y = (\cos x)^{\sin x}$

$\ln y = \ln (\cos x)^{\sin x}$ **rule: $\ln A^n = n \ln A$**

$\ln y = \sin x \ln(\cos x)$

$\frac{y`}{y} = (\sin x)` \ln(\cos x) + [\ln(\cos x)]` \sin x$ **Take heed: ln (cosx) is a derivative of complex function.**

$\frac{y`}{y} = \cos x \ln(\cos x) + \frac{1}{\cos x}(\cos x)` \sin x$

$\frac{y`}{y} = \cos x \ln(\cos x) + \frac{1}{\cos x}(-\sin x) \sin x$ **simplify little…**

$\frac{y`}{y} = \cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x}$ **all multiply with y**

$y` = y [\cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x}]$ **replace:** $y = (\cos x)^{\sin x}$

$y` = (\cos x)^{\sin x}[\cos x \ln(\cos x) - \frac{\sin^2 x}{\cos x}]$ is solution.

Example 3.

Find derivative of function: $y = \left(\frac{\ln x}{x}\right)^{\sin x}$

Solution:

$$y = \left(\frac{\ln x}{x}\right)^{\sin x}$$

$$\ln y = \ln\left(\frac{\ln x}{x}\right)^{\sin x}$$

$$\ln y = \sin x \ \ln\left(\frac{\ln x}{x}\right)$$

$$\frac{y`}{y} = (\sin x)`\ln\left(\frac{\ln x}{x}\right) + [\ln\left(\frac{\ln x}{x}\right)]` \sin x$$

$$\frac{y`}{y} = \cos x \ \ln\left(\frac{\ln x}{x}\right) + \frac{1}{\frac{\ln x}{x}}\left(\frac{\ln x}{x}\right)` \sin x$$

$$\frac{y`}{y} = \cos x \ \ln\left(\frac{\ln x}{x}\right) + \frac{x}{\ln x}\,\frac{(\frac{1}{x}x - \ln x)}{x^2} \sin x$$

$$\frac{y`}{y} = \cos x \ \ln\left(\frac{\ln x}{x}\right) + \frac{x}{\ln x}\,\frac{(1 - \ln x)}{x^2} \sin x$$

$$\frac{y`}{y} = \cos x \ \ln\left(\frac{\ln x}{x}\right) + \frac{(1 - \ln x)}{x \ln x} \sin x$$ **all multiply with y**

$$y` = y\,[\cos x \ \ln\left(\frac{\ln x}{x}\right) + \frac{(1 - \ln x)}{x \ln x} \sin x]$$ **replace:** $y = \left(\frac{\ln x}{x}\right)^{\sin x}$

$$\mathbf{y`} = \left(\frac{\ln x}{x}\right)^{\sin x} [\cos x \ \ln\left(\frac{\ln x}{x}\right) + \frac{(1 - \ln x)}{x \ln x} \sin x]$$ **the final solution!**

Derivatives of function given in Parametric form

If in function y = f (x) variable x and y depend on the parameters t (x=x(t) and y=y(t)) , the first derivative of function y = f (x) is calculate by the formula:

$$y_x^` = \frac{y_t^`}{x_t^`}$$

Example 1.

Calculate the first derivative function given in parametric form: $x = 2t - t^2$ and $y = 4t - t^3$

Solution:

$\mathbf{x = 2t - t^2}$ from here is $x`_t = 2 - 2t$
$\mathbf{y = 4t - t^3}$ from here is $y`_t = 4 - 3t^2$ Now $x`_t$ and $y`_t$ add to the formula:

$$y_x^` = \frac{y_t^`}{x_t^`} = \frac{2-2t}{4-3t^2}$$ and here is solution!

Example 2.

Calculate the first derivative function given in parametric form: x = r cost and y = r sint

Solution:

x = r cost ⟶ $x`_t$ = - r sin t

y = r sint ⟶ $y`_t$ = r cost

$$y_x^` = \frac{y_t^`}{x_t^`} = \frac{-r\cos t}{r\sin t} = \frac{-\cos t}{\sin t} = -\text{ctg } t$$

Example 3.

Calculate the first derivative function given in parametric form: x = cost +t sint and y = sint – t cost

Solution:

x = cost +t sint from here is: $x`_t$ = - sint + [ t`sint + (sint)`t] = - sint + sint + t cost = t cost

y = sint – t cost so: $y`_t$ = cost – [t` cost + (cost)` t] = cost – cost + t sint = t sint

$$y_x^` = \frac{y_t^`}{x_t^`} = \frac{t\sin t}{t\cos t} = \frac{\sin t}{\cos t} = \text{tg } t$$

Derivatives of function given in Implicit form

When the function y = f (x) is given in implicit form F (x, y) = 0, its first derivative obtained from the relation:

$$\frac{d}{dx}F(x,y)=0$$

Example 1.

Find derivative of function: $x^3 - 2y - y^2 = 0$

Solution:

Mark: $F(x,y) = x^3 - 2y - y^2$

$\frac{d}{dx}F(x,y) = 3x^2 - 2y` - 2yy`$ now this equal with 0

$3x^2 - 2y` - 2yy` = 0$ from here now express y` , and that's it.

$3x^2 = 2y` + 2yy`$

$3x^2 = 2y`(1+y)$ then **$y` = \frac{3x^2}{2(1+y)}$ is final solution.**

Example 2.

Find derivative of function: $\mathbf{x^2 + xy + y^2 + 6 = 0}$

Solution:

$F(x,y) = x^2 + xy + y^2 + 6$

$\frac{d}{dx}F(x,y) = 2x + y + xy` + 2yy`$

$2x + y + xy` + 2yy` = 0$ from here we need to express y`

$xy` + 2yy` = -2x - y$

$y`(x + 2y) = -2x - y \longrightarrow y` = \frac{-2x-y}{x+2y}$ **is final solution.**

Example 3.

Calculate the first derivative of function : $e^{xy} = x^3 - y^3$

Solution:

$e^{xy} = x^3 - y^3$

$e^{xy}(xy)` = 3x^2 - 3y^2y`$

$e^{xy}(y + xy`) = 3x^2 - 3y^2y`$

$e^{xy}y + e^{xy}\, xy` = 3x^2 - 3y^2y`$

$e^{xy}\, xy` + 3y^2y` = 3x^2 - e^{xy}y$

$y`( e^{xy}\, x + 3y^2) = 3x^2 - e^{xy}y$ **express** y`

$y` = \frac{3x^2 - e^{xy}y}{e^{xy}x + 3y^2}$ is final solution.

Example 4.

Calculate the first derivative of function : $x^y - y^x = 0$

Solution:

$x^y = y^x$

$\ln x^y = \ln y^x$

$y \ln x = x \ln y$ derivative…

$y`\ln x + y\frac{1}{x} = \ln y + \frac{1}{y}y`x$

$y` \ln x - \frac{x}{y}y` = \ln y - \frac{y}{x}$

$y` (\ln x - \frac{x}{y}) = \ln y - \frac{y}{x}$

$$y` = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$$

Derivatives of inverse function

Some function **f** has the first derivative different from 0, and function **g** is its inverse function. Then function

g also has derivative and:

$$g`(x) = \frac{1}{f`(g(x))}$$

Often, this formula is written in the form of:

$$y_x^{`} = \frac{1}{x_y^{`}} \qquad \text{or} \qquad x`_y\, y`_x = 1$$

Example 1.

If y = arcsin x , -1 ≤ x ≤ 1, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ **, find derivative from y !**

Solution:

Since y = arcsin x , then x = siny , we use $y_x^{`} = \frac{1}{x_y^{`}}$ and we have :

$$(\arcsin x)` = \frac{1}{(\sin y)`} = \frac{1}{\cos y} = \{\text{ Now use that: } \sin^2 y + \cos^2 y = 1 \text{ or } \cos y = \sqrt{1-\sin^2 y}\ \} =$$

$$= \frac{1}{\sqrt{1-\sin^2 y}} = \{\text{ replace } \sin y = x\ \} = \frac{1}{\sqrt{1-x^2}}$$

So: $(\arcsin x)` = \frac{1}{\sqrt{1-x^2}}$

Example 2.

If y = arctg x and $-\infty < x < \infty$ **and** $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ **find derivative from y !**

Solution:

Since y = arctg x it must be x = tg y , and apply $y_x^` = \frac{1}{x_y^`}$:

$$(\text{arctg}x)^` = \frac{1}{(tgy)^`} = \frac{1}{\frac{1}{\cos^2 y}} = \{\sin^2 y + \cos^2 y = 1\}$$

$$= \frac{1}{\frac{\sin^2 y + \cos^2 y}{\cos^2 y}} = \frac{1}{\frac{\sin^2 y}{\cos^2 y} + \frac{\cos^2 y}{\cos^2 y}} = \frac{1}{1 + tg^2 y}$$ **replace: tgy = x**

$$= \frac{1}{1 + x^2}$$

So: $(\text{arctg}x)^` = \frac{1}{1+x^2}$

Example 3.

If y = log$_a$x , a>0 , a $\neq$ **1 , x>0 and** $-\infty < y < \infty$ **, find derivative.**

Solution:

Inverse function for y = log$_a$x is x = a^y, and by the formula $y_x^` = \frac{1}{x_y^`}$ is:

$$(\log_a x)^` = \frac{1}{(a^y)^`}$$

$$= \frac{1}{a^y \ln a}$$ = and now only replace x = a^y

$$= \frac{1}{x \ln a}$$

So: $(\log_a x)^` = \frac{1}{x \ln a}$